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On certain Methods of dividing the Surplus among the Assured in a Life Assurance Company; and on the Rates of Premium that should be charged to render them equitable. By T. B. SPRAGUE, M.A., *Fellow of St. John's College, Cambridge.*

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THE subject of the division of surplus in a Life Assurance Company is one that cannot fail to be interesting to all who are engaged in the business of life assurance, as well as to the public at large; and while there is so great a diversity as at present exists in the methods of division in use, it must be useful occasionally to draw attention to it, as a means of inviting discussion and the interchange of opinions. This must be my apology for bringing before this Society this evening a subject which has more than once been discussed here; for I can scarcely hope to lay before you in my remarks much that has the merit of novelty, on a matter that has occupied the attention of the most eminent men in our profession.

The methods of division in use may be arranged, with sufficient accuracy for our present purpose, into three classes: the first containing those methods which give rise to a series of nearly constant cash bonuses on each policy; the second, those by which the amounts of the successive reversionary bonuses on each policy will be nearly constant; and the last including the various methods which lead to a series of reversionary bonuses increasing with

greater or less rapidity. This classification is only made for purposes of convenience, and is not by any means put forward as embracing all the methods now employed.

Under the first class will come the methods according to which the surplus is divided at the periodical valuation, in proportion to the amount of the premiums paid since the previous valuation, or in proportion to the amount of the loadings of those premiums. As an instance of the second class, we may take the method that adds to all the policies at each valuation the same percentage per annum for the number of years the policy has been in force since the previous valuation. The last class will include a variety of methods—that, for instance, which divides the surplus in proportion to the excess of the premiums paid over the value of the policy; as also those which make the division in proportion to the value of the policy, and in proportion to the amount of the premiums paid on the policy from the commencement; and lastly, that method which adds to every policy at each valuation a uniform percentage per annum for the whole number of years the policy has been in force.

Of all the methods above enumerated, that which appears to be the most equitable to all parties, is the one proposed by Mr. Jellicoe, according to which the surplus is divided in proportion to the amount of the loading of the premium charged, accumulated at compound interest for the interval between successive valuations. For, it is argued, suppose that the *premium for the risk* has been determined with as much accuracy as circumstances admit, having regard to the rate of mortality and the rate of interest prevailing; then the excess of the premium charged over this net premium, is the chief source of the surplus shown by the valuation of a Company's business; and it is certainly equitable to return the surplus to the assured in the ratios in which they have respectively contributed to form it. But this method cannot be considered as rigorously equitable, for the surplus arises from other sources besides that indicated: it is only, in fact, a very good approximation, and this is all that we can ever expect to attain. The method under consideration, however, has the merit of doing justice to all parties, by giving them a bonus regulated immediately by the premiums which they pay; and in particular, of making a distinction between that part of the premium which is necessary to provide for the sum originally assured, and the loading, or the part that provides for the bonuses. Most of the methods enumerated above proceed upon very vague

notions of what is just to the assured; and some of them seem to have been adopted simply because they are convenient, regardless of justice. But it is observable, that they all have very little reference, if any, to the manner in which the premiums charged are derived from the net premiums; whereas it must be admitted that it is quite impossible to do justice, without considering the two subjects of the loading of the net premium, and the method of division of profits, in the very closest connection.

Although the method above specified is as equitable as any that could be devised, there seems to be one point in its working which is open to some objection: I mean, that since the successive cash bonuses on each policy will be of about the same magnitude, supposing the state of the Company to remain the same, it follows, that the successive reversionary bonuses will form a decreasing series, on account of the increasing age of the life assured. This, of course, is no objection to the theory of the method; but I think it will be admitted, that the public at large, who understand nothing of the theory of life assurance, expect that each reversionary bonus on their policies should be at least as large as the preceding one; and it will probably be thought an advantage, if a method of division could be adopted which while equitable to all parties, and simple in its details, should at the same time satisfy the condition mentioned above. It is my purpose to-night to indicate a method by which it seems these objects may be to some extent attained.

The first idea that suggests itself, is to distribute only a portion of the surplus that appears as the result of the valuation, and reserve the remainder to accumulate, and to be divided at a future time among the survivors of those who have contributed towards it. If this condition is strictly adhered to, and no part of the reserved surplus is divided among those who have not contributed towards it, there can be little doubt, I think, that the method of division would be equitable; and the object in view, of obtaining a series of constant or increasing bonuses, might be thereby attained. But this method would introduce complications that would probably be found inconvenient in practice; and it will be desirable to adopt a plan that shall virtually involve the same principle, but be simpler in its details. I propose to seek such a plan in a modification of one already in use, and to which I have referred above—namely, the method according to which at each valuation an addition is made to every policy of a uniform percentage per annum on the sum assured, for the number of years it has been in force since the previous valua-

tion. It is very commonly the case, that the percentage is added, not only on the sums originally insured, but also on the previous additions—on what principle of equity, it is not easy to imagine, but probably only with the view of providing that the bonuses shall gradually increase in magnitude on each policy. Now the correct way to look upon this method of division seems to be, to consider the bonuses to be added to a policy as forming an increasing assurance for which a constant annual premium is charged: this premium for the bonuses being in fact a portion of the loading or addition to the net premium, while the remainder of this loading provides for the expenses of management. On this supposition, it may be shown that the method practically adopted of giving a uniform rate of increase without regard to age—though it may be equitable as between persons insuring at the same age, but at different periods of the existence of the Company—is not, on the usual suppositions, equitable as between persons insuring at different ages. In order to show this, I will suppose that the net premiums, as is very commonly the case, are loaded with a constant percentage, and proceed to investigate the amount of the increasing assurance that this loading will provide.

Thus, let P_n = net annual premium to insure £100, P_n' = premium charged, $\pi_n = P_n' - P_n$ = loading. Then, with the common notation, the value of an increasing assurance of £k per annum is $k \cdot \dfrac{M_{n+1}+M_{n+2}+\cdots}{D_n}$ or $k \cdot \dfrac{R_{n+1}}{D_n}$, and the annual premium for the same is $k \cdot \dfrac{R_{n+1}}{N_{n-1}}$. But, by the hypothesis, the loading π_n is the annual premium for this increasing assurance;

so that
$$\pi_n = k \cdot \frac{R_{n+1}}{N_{n-1}} \quad . \quad . \quad . \quad . \quad (1)$$

But $P_n = \dfrac{M_n}{N_{n-1}}$; whence $\dfrac{\pi_n}{P_n} = k \cdot \dfrac{R_{n+1}}{M_n}$,

and
$$k = \frac{M_n}{R_{n+1}} \cdot \frac{\pi_n}{P_n} \quad . \quad . \quad . \quad (2)$$

In order to obtain a numerical result from this equation, we must assume some relation between π_n and P_n. I take $\dfrac{\pi_n}{P_n} = \dfrac{3}{10}$ for the following reason:—the premiums charged by some of the existing Insurance Companies are based apparently on the Carlisle 4 per Cent. net premium, with an addition of 40 per cent.; one

fourth of this loading will be a fair deduction for expenses, leaving three fourths applicable for providing the bonus; which leads at once to the above equation, $\frac{\pi_n}{P_n}=\frac{3}{10}$. Then, calculating the value of $\frac{M_n}{R_{n+1}}$ from the Carlisle 4 per Cent. Tables,* for the ages 25, 35, 45, we arrive at the following results:—

When $n=25$, we have $k=1{\cdot}20$
" $n=35$, " $k=1{\cdot}37$
" $n=45$, " $k=1{\cdot}63$

It appears, then, that in the case of assurances taken out at the age of 25, the above loading will provide for a bonus addition at the rate of 1·2 per cent. per annum; at age 35 the rate will be 1·37, and at age 45 it is 1·63.

It results from this investigation, that the common method of adding a uniform percentage on all the sums assured, without regard to age, has the effect, when the premiums charged are formed by adding a percentage to the net premium, of giving too much bonus to those who assure at the younger ages, and too little to those who assure at more advanced ages.

The formula given above is only *strictly* applicable in the case of an annual division of profits; but when there is an interval bonus allowed of the same, or nearly the same, magnitude as that declared at the periodical valuations, the formula will represent the facts of the case with sufficient accuracy for all practical purposes.

If, instead of adding a percentage to the net premium, we suppose the premiums charged to be obtained by making a *constant addition* to the net premiums at every age, we may prove by means of the equation (2) that the common method of giving a bonus addition at a uniform rate on the sum assured, without regard to age, gives too small a bonus to the younger lives, and too large a bonus to the older lives.

It appears, then, that when the premiums charged are formed in the usual way, the very convenient practical course of adding a uniform percentage to all the sums assured, which combines the merit of simplicity with economy of time and labour, leads to results that are more or less inequitable. The correct additions to be made on these suppositions would have to be determined by calculations more or less complicated, which it is no part of my

* I take the rate of interest at 4 per cent., because this is the rate that seems to be generally realized by Insurance Companies on the bulk of their investments. I should have preferred using the Experience Table of Mortality, but have not access to any tables on the Experience 4 per Cent. basis which contain the M and R columns.

purpose to consider in this paper: some of the formulæ for this case have been investigated by Mr. Wilbraham, in a recent number of the *Assurance Magazine*. I propose to secure the same object, of returning to each of the assured a fair equivalent for the premium paid, by graduating the premiums charged so as to correspond with the method of division of profits; or, in other words, to add to the net premium, which provides for the sum originally assured, an annual premium that shall provide for the increasing assurance formed by the bonuses. If, as will appear presently, premiums so formed should not differ much from those at present in use, we should be able to adopt, with strict justice towards all the assured, the simple, convenient, and popular method of division of profits which adds the same percentage to all the sums assured.

The formula (1) above will enable us to calculate the additions to the premiums. Thus, in order to find the annual premium to provide for £100 at death, and a bonus addition at the rate of $1\frac{1}{2}$ per cent. per annum, we must make $k=1{\cdot}5$ in formula (1); and for a bonus addition at the rate of 2 per cent. we must make $k=2$. Using, as before, the Carlisle 4 per Cent. Tables, we arrive at the premiums in the following table:—

Age.	P_1.	P_2.	P_3.
	£. s. d.	£. s. d.	£. s. d.
25	1 10 4	2 1 9	2 5 6
35	2 0 5	2 13 8	2 18 1
45	2 15 6	3 10 10	3 15 11

Here P_1 = net annual premium to insure £100 at death; P_2 = annual premium to insure £100, and a bonus addition at the rate of $1\frac{1}{2}$ per cent. per annum; P_3 = the same, but with a bonus addition at 2 per cent. per annum.

These premiums have no addition for expenses. An addition of 10 per cent. will be moderate, and probably sufficient, since the bonus is not guaranteed, and will not be declared unless the state of the Office admit of it. This addition is therefore made below, to the premium P_2; and some of the premiums now in use are given at the same time, for the purpose of comparison.

Age.	$P_2+\frac{P_2}{10}$.	Northampton.	Carlisle.
	£. s. d.	£. s. d.	£. s. d.
25	2 5 11	2 8 1	2 2 6
35	2 19 0	2 19 10	2 16 8
45	3 17 11	3 17 11	3 17 8

In this table the premiums under the heading "Northampton" are the old Northampton rates, the use of which was formerly all but universal, and which are still used by some of the Offices. Those under the heading "Carlisle" are the premiums already alluded to, which are deduced from the Carlisle 4 per Cent. Net Premium by adding 40 per cent. It will be seen that the rates obtained on the above suppositions do not differ very much from some of those at present in use.

If there is a body of proprietors in the Company, to whom one fifth of the profits are allotted at each valuation, we must still further increase the premium. In this case we must add to the net premium $\frac{5}{4}\pi_n$, instead of π_n as given by equation (1). The result of this will be, that instead of the premiums under the heading "P_2" in the former of the preceding tables, we shall have the following, in which, as before, no addition has been made for expenses:—at age 25, £2. 4*s*. 7*d*.; at 35, £2. 17*s*.; at 45, £3. 14*s*. 8*d*.

The process employed in the preceding investigation may be applied to other cases besides the above, and the correct premium ascertained when the method of division is different from the one just considered. Thus, for example: take that method according to which at the periodical valuations a percentage is added to a policy, not only on the sum assured, but also on the previous additions. Suppose that the additions are made at the rate of £k per annum for every £1 assured, and that the valuations are quinquennial, then a policy for £1 will have k added to it at the end of each of the first five years, and the total amount at the end of five years will be $1+5k$. At the end of each of the next five years an addition will therefore be made of $k(1+5k)$; the amount of these additions is $5k(1+5k)$, and the total amount of the policy is $1+5k+5k(1+5k)$ or $(1+5k)^2$. In the same way, at the end of each of the next five years an addition will be made of $k(1+5k)^2$; and at the end of five years the total amount will be $(1+5k)^3$, and so on for subsequent periods—the law that these expressions follow being manifest. Next to find the present value of these additions: the value of k, to be added to the policy at the end of each of the first five years, will be

$$k\frac{M_{n+1}+M_{n+2}\ldots+M_{n+5}}{D_n}=k\frac{R_{n+1}-R_{n+6}}{D_n}.$$

The value of $k(1+5k)$, to be added during each of the next five years, will be

$$k(1+5k)\frac{M_{n+6}+M_{n+7}+\ldots+M_{n+10}}{D_n}=k(1+5k)\frac{R_{n+6}-R_{n+11}}{D_n},$$

and similarly for subsequent additions. Hence the value of all the additions will be

$$k\frac{R_{n+1}-R_{n+6}}{D_n}+k(1+5k)\frac{R_{n+6}-R_{n+11}}{D_n}+k(1+5k)^2\frac{R_{n+11}-R_{n+16}}{D_n}+\text{\&c.}$$

or

$$\frac{1}{D_n}\left\{kR_{n+1}+5k^2R_{n+6}+5k^2(1+5k)R_{n+11}+5k^2(1+5k)^2R_{n+16}+\text{\&c.}\right\}=\frac{\Sigma}{D_n}$$

suppose.

Hence the annual premium for these additions will be $\frac{\Sigma}{N_{n-1}}$. The premiums, as calculated by this formula, to insure £100, with additions at the rate of $1\frac{1}{2}$ per cent. per annum on the sum assured and on previous additions, are, at age 25, £2. 5*s.*; at 35, £2. 16*s.* 9*d.*; and at 45, £3. 13*s.* 7*d.*: being about 3*s.* higher than the rates required when the additions are not made upon previous additions.

Again, take the method according to which at each valuation a percentage is added to the sum assured for the whole number of years it has been in force. The formula for the premium to provide for the bonus additions in this case, at the rate of £*k* for every £100 assured, is easily seen to be

$$\pi_n=5k\,.\,\frac{M_{n+5}+2M_{n+10}+3M_{n+15}+\ \ldots\ldots}{N_{n-1}}\,.$$

It is supposed here that there is no prospective bonus.

The premiums have been calculated, as before, for the three ages, 25, 35, 45, and the rate of addition has been taken at $1\frac{1}{2}$ per cent. per annum. The results are given in the following table:—

Age.	25.	35.	45.
	£. s. d.	£. s. d.	£. s. d.
Premium to insure £100	1 10 4	2 0 5	2 15 6
Premium to insure the additions.	2 3 3	2 3 1	2 0 10
Total	£3 13 7	£4 3 6	£4 16 4

It will be seen, that for the younger ages, the premium to provide for the additions to the policy is larger than that required to provide for the original sum, and the whole premium to provide for the original sum and the additions is an extremely high one;

though, as will be remembered, no addition has been made as yet to the premium for the expenses of conducting the Office. It follows from this result, that when this method of division is adopted, a rate of addition so high as $1\frac{1}{2}$ per cent. cannot be maintained. By way of ascertaining what rate of addition may be looked for in the long run, I have calculated the value of k on the same supposition as before, that the part of the premium applicable to profits is three tenths of the net premium, or $\frac{\pi_n}{P_n}=\frac{3}{10}$. The formula for k is readily seen to be

$$k=\frac{1}{5}\frac{\pi_n}{P_n}\frac{M_n}{M_{n+5}+2M_{n+10}+3M_{n+15}+\ldots}.$$

The result is, at the age 35, $k=\cdot423$; so that, instead of an addition at the rate of $1\frac{1}{2}$, we cannot expect a permanent addition so high as $\frac{1}{2}$ per cent., unless the profits of the Company, from some sources not taken account of here, should be very large, such as from unexpected improvement in the value of investments. Of course, if the number of the assured who are allowed to participate in the profits is limited, the amount of the additions given to the favoured few may be considerably greater than that calculated above.

I shall conclude this paper with a few remarks on the nature of the valuations in an Office proceeding on the suppositions here laid down. We suppose, then, that the Office charges rates which will provide for an addition to the policy at the rate of $1\frac{1}{2}$ per cent. per annum, and engages to give the assured additions as nearly of this magnitude as the state of the Office will allow. The first consideration that suggests itself is, that since the same addition, or nearly so, is to be made to the sum assured at each valuation, and the age of the life is continually increasing, the value of this addition will increase at each successive valuation; but since the profit arising from each policy is nearly the same for the period between any two valuations, it will be necessary, in order to meet the continually increasing charge on the Office, to divide only a portion of what appears as the available surplus, and reserve the remainder to accumulate. If this precaution is neglected, and the whole amount that appears from the result of the valuation to be divisible surplus is divided at the time, it may easily happen, that at several successive valuations in the earlier stages of the existence of the Office, a percentage may be given to the assured larger than that provided for by the premium. The effect however

will be, unless the profits of the Office from some extraneous sources, such as lapsed or surrendered policies, or improvement in the value of investments, are considerable enough to compensate, that the percentage to be distributed on future occasions will have to be considerably reduced, and probably fall below that allowed for in the premiums. This effect is scarcely to be looked for during the first twenty or thirty years of the existence of an Office, especially if it receives a gradually increasing number of new members each year; for it is shown, in a very ingenious paper by Dr. Farr, that, supposing the number of new members that enter an Office to be the same each year, a period of sixty years will elapse before the Office attains its full maturity, or stationary period, in which the number of deaths each year is exactly equal to the number of new members who enter, and the outgoings from claims exactly equal to the income from premiums and interest. The effects just mentioned will be still more manifest, and probably occur sooner, in an Office which divides its surplus according to the other method considered, and adds at each valuation a percentage for the whole number of years the policy has been in force.

In order to guard against such a result, which would of course produce great disappointment among the assured, it will be necessary to have recourse to some such process as the following in making the periodical investigation into the state of the Office preparatory to declaring a bonus:—

The valuation of the liabilities of the Office having been made, and credit taken only for the future net premiums, we must ascertain whether the available surplus will provide for an addition to all the participating policies at the rate of $1\frac{1}{2}$ per cent. per annum, for the number of years elapsed since the previous valuation. If the Office has been prosperous, the available surplus should be more than enough for this purpose; and, after providing for these additions, leave unappropriated an amount, R, suppose. We must then calculate the value of the future additions to the policies at the rate of $1\frac{1}{2}$ per cent. per annum, which may be done by the ordinary formula for an increasing assurance. On the other side of the account, we take credit for the loadings of the premiums to be received on the existing policies. We will call these respective amounts B and L: then, if it should appear that $B > L + R$, the consequence follows that the funds in hand and the future premiums are insufficient to provide for the anticipated rate of addition to the policies; and if the deficiency is large, it will be a

question requiring attentive consideration whether the additions to be made at the present time should be of the full amount of $1\frac{1}{2}$ per cent. On the other hand, if $L+R>B$, the assets of the Office, present and prospective, are more than sufficient to make the proposed additions; and if the excess is considerable, it may be prudent to declare a bonus at the present time at a higher rate than $1\frac{1}{2}$ per cent. In this process no account has been taken of the part of the premium added to provide for expenses; this will render some modification of the above necessary in practice.

On the Use of the Integral Calculus in determining Averages, with certain Applications to the Theory of Life Contingencies. By SAMUEL YOUNGER, *of the Engineers' Life Office.*

IN a paper* which was read before the Institute of Actuaries on the 8th January, 1855, the idea was first suggested by Mr. Farren of introducing into the theory of life contingencies the hypothesis of a constantly fluctuating rate of interest, in lieu of the supposition hitherto adopted of a uniform rate.

Abundant reasons are assigned, in the paper referred to, for making the proposed change; and no one, at all acquainted with the practice of life assurance, can fail to see that the effect of Mr. Farren's suggestion, if carried out, will be to base all computations upon an assumption which is in the closest possible accordance with actual facts.

That opposition, in some quarters, will be displayed to any innovation upon existing methods and formulæ, is tolerably certain. It will be maintained, perhaps, that while the proposed amendment will not very materially alter numerical results obtainable by the present mode of calculation, it will entail much additional trouble in finding them; that the analytical formulæ will be too complex to be of practical utility; that, in short, the whole existing fabric will be destroyed, and every actuary compelled to begin his studies again. The main object of the present paper is to show the fallacy of these objections, by exhibiting the results which flow from the assumption of an ever varying rate of interest—first, in the case of life annuities, and then in the case of life assurances. It will be seen, that if a new table of annuities

* *On the Improvement of Life Contingency Calculation.* By Edwin James Farren, Esq. See *Assurance Magazine*, vol. v., pp. 185, &c.